Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10% ?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

Answer (B): Because $\frac{x+1}{y+1} = \frac{11}{10} \cdot \frac{x}{y}$, it follows that $10y - 11x - xy = 0$ and so $(10 - x)(11 + y) = 110 = 2 \cdot 5 \cdot 11$. The only possible values of $10 - x$ are 5, 2, and 1 because x and y are positive integers. Thus the possible values of x are 5, 8, and 9. Of the resulting fractions $\frac{5}{11}$, $\frac{8}{44}$, and $\frac{9}{99}$, only the first is in simplest terms.

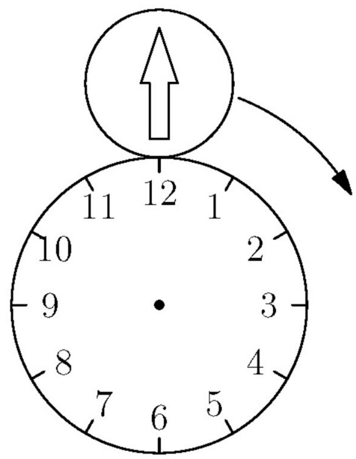
If $y + 4 = (x - 2)^2$, $x + 4 = (y - 2)^2$, and $x \neq y$, what is the value of $x^2 + y^2$?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

Answer (B): Expanding the binomials and subtracting the equations yields $x^2 - y^2 = 3(x - y)$. Because $x - y \neq 0$, it follows that $x + y = 3$. Adding the equations gives $x^2 + y^2 = 5(x + y) = 5 \cdot 3 = 15$.

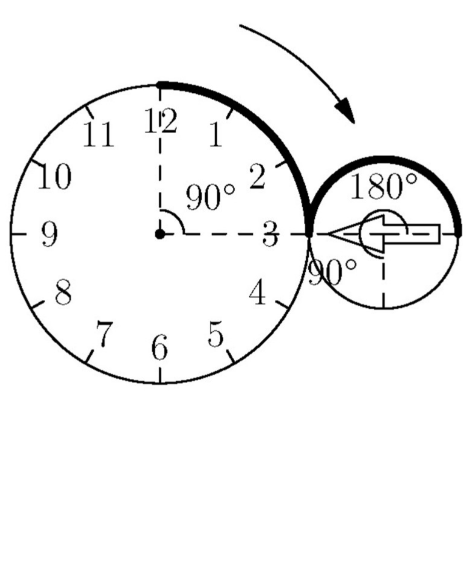
Note: The two solutions are $(x, y) = (\frac{3}{2} + \frac{\sqrt{21}}{2}, \frac{3}{2} - \frac{\sqrt{21}}{2})$ and $(\frac{3}{2} - \frac{\sqrt{21}}{2}, \frac{3}{2} + \frac{\sqrt{21}}{2})$.

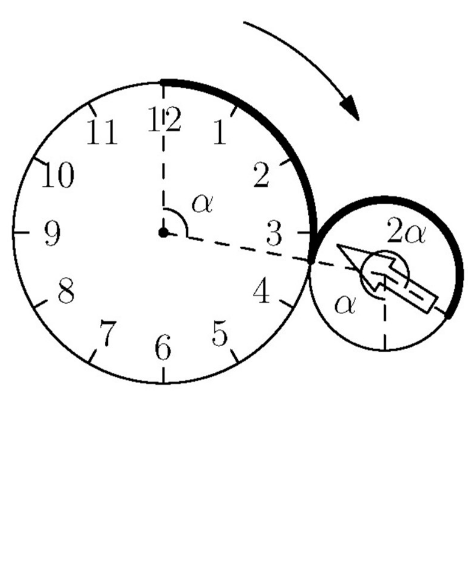
The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?



(A) 2 o'clock **(B)** 3 o'clock **(C)** 4 o'clock **(D)** 6 o'clock **(E)** 8 o'clock

Answer (C): The circumference of the disk is half the circumference of the clock face. As the disk rolls $\frac{1}{4}$ of the way around the circumference of the clock face (from 12 o'clock to 3 o'clock), the disk rolls through $\frac{1}{2}$ of its own circumference. At that point, the arrow of the disk is pointing at the point of tangency, so the arrow on the disk will have turned $\frac{3}{4}$ of one revolution. In general, as the disk rolls through an angle α around the clock face, the arrow on the disk turns through an angle 3α on the disk. The arrow will again be pointing in the upward vertical direction when the disk has turned through 1 complete revolution, and the angle traversed on the clock face is $\frac{1}{3}$ of the way around the face. The point of tangency will be at 4 o'clock.





Hexadecimal (base-16) numbers are written using the numeric digits 0 through 9 as well as the letters A through F to represent 10 through 15. Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Answer (E): Because $1000 = 3 \cdot 16^2 + 14 \cdot 16 + 8$, the largest number less than 1000 whose hexadecimal representation contains only numeric digits is $3 \cdot 16^2 + 9 \cdot 16 + 9$. Thus the number of such positive integers is $n = 4 \cdot 10 \cdot 10 - 1 = 399$ ($0 \cdot 16^2 + 0 \cdot 16 + 0 = 0$ is excluded), and the sum of the digits of n is 21.

The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$?

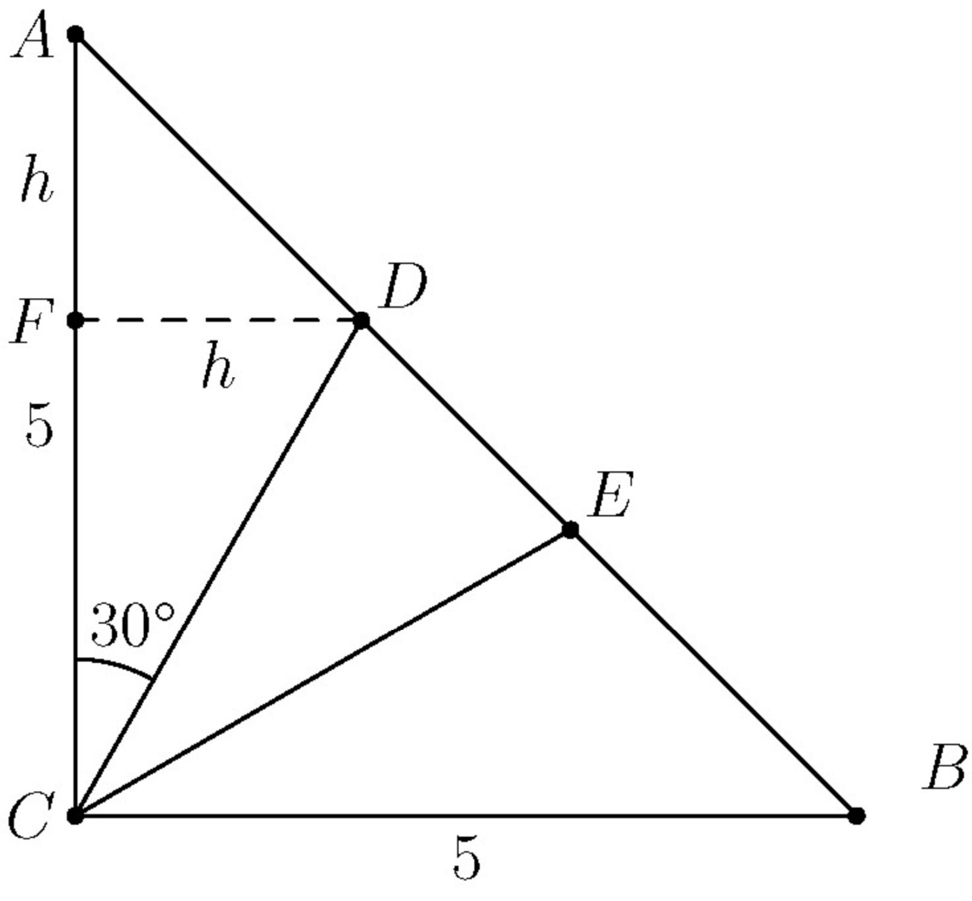
(A) $\dfrac{5\sqrt{2}}{3}$ **(B)** $\dfrac{50\sqrt{3} - 75}{4}$ **(C)** $\dfrac{15\sqrt{3}}{8}$ **(D)** $\dfrac{50 - 25\sqrt{3}}{2}$ **(E)** $\dfrac{25}{6}$

Answer (D): Because the area is 12.5, it follows that $AC = BC = 5$. Label D and E so that D is closer to A than to B. Let F be the foot of the perpendicular to $\overline{AC}$ passing through D. Let $h = FD$. Then $AF = h$ because $\triangle ADF$ is an isosceles right triangle, and $CF = h\sqrt{3}$ because $\triangle CDF$ is a $30-60-90^\circ$ triangle. So $h + h\sqrt{3} = AC = 5$ and

$$h = \frac{5}{1+\sqrt{3}} = \frac{5\sqrt{3}-5}{2}.$$

Thus the area of $\triangle CDE$ is

$$\frac{25}{2} - 2 \cdot \frac{1}{2} \cdot 5 \cdot \frac{5\sqrt{3}-5}{2} = \frac{50-25\sqrt{3}}{2}.$$



A rectangle has area A cm^2 and perimeter P cm, where A and P are positive integers. Which of the following numbers cannot equal $A + P$?

(A) 100 **(B)** 102 **(C)** 104 **(D)** 106 **(E)** 108

Answer (B): Let x and y be the lengths of the sides of the rectangle. Then $A+P = xy+2x+2y = (x+2)(y+2)-4$, so $A+P+4$ must be the product of two factors, each of which is greater than 2. Because the only factorization of $102+4=106$ into two factors greater than 1 is $2 \cdot 53$, $A+P$ cannot equal 102. Because $100+4=104=4 \cdot 26$, $104+4=108=3 \cdot 36$, $106+4=110=5 \cdot 22$, and $108+4=112=4 \cdot 28$, the other choices equal $A+P$ for rectangles with dimensions 2×24, 1×34, 3×20, and 2×26, respectively.

Among the positive integers less than 100, each of whose digits is a prime number, one is selected at random. What is the probability that the selected number is prime?

(A) $\frac{8}{99}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{9}{20}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{9}{16}$

Answer (B): There are four one-digit primes (2, 3, 5, and 7), which can be used to form $4^2 = 16$ two-digit numbers with prime digits. Of these two-digit numbers, only 23, 37, 53, and 73 are prime. So there are $4+16=20$ numbers less than 100 whose digits are prime, and $4+4=8$ of them are prime. The probability is $\frac{8}{20} = \frac{2}{5}$.

Let a, b, and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation $(x-a)(x-b)+(x-b)(x-c)=0$?

(A) 15 **(B)** 15.5 **(C)** 16 **(D)** 16.5 **(E)** 17

Answer (D): If $(x-a)(x-b)+(x-b)(x-c)=0$, then $(x-b)(2x-(a+c))=0$, so the two roots are b and $\frac{a+c}{2}$. The maximum value of their sum is $9+\frac{8+7}{2}=16.5$.

The town of Hamlet has 3 people for each horse, 4 sheep for each cow, and 3 ducks for each person. Which of the following could not possibly be the total number of people, horses, sheep, cows, and ducks in Hamlet?

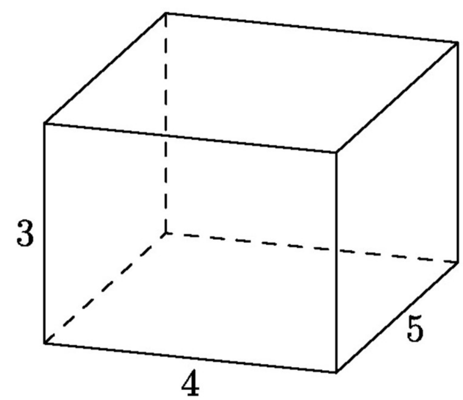
(A) 41 **(B)** 47 **(C)** 59 **(D)** 61 **(E)** 66

Answer (B): Let h be the number of horses and c be the number of cows. There are then $3h$ people, $9h$ ducks, and $4c$ sheep in Hamlet. The total population of Hamlet is $13h + 5c$, where h and c are whole numbers. A number N can be the population only if there exists a whole number value for h such that $N - 13h$ is a whole number multiple of 5. This is possible for all the provided numbers except 47, as follows: $41 - 13 \cdot 2 = 5 \cdot 3$, $59 - 13 \cdot 3 = 5 \cdot 4$, $61 - 13 \cdot 2 = 5 \cdot 7$, and $66 - 13 \cdot 2 = 5 \cdot 8$

None of 47, $47 - 13 = 34$, $47 - 13 \cdot 2 = 21$, and $47 - 13 \cdot 3 = 8$ is a multiple of 5. Therefore 47 cannot be the population of Hamlet.

Note: In fact, 47 is the largest number that cannot be the population.

The centers of the faces of the right rectangular prism shown below are joined to create an octahedron. What is the volume of the octahedron?



(A) $\frac{75}{12}$ **(B)** 10 **(C)** 12 **(D)** $10\sqrt{2}$ **(E)** 15

Answer (B): Consider the octahedron to be two pyramids whose base is a rhombus in the middle horizontal plane, as shown below. One pyramid points up, the other down. The area of the base is the area of 4 right triangles with legs 2 and $\frac{5}{2}$, or 10. The altitude of each pyramid is half that of the prism or $\frac{3}{2}$. The volume of the octahedron is $2 \cdot \frac{1}{3} \cdot 10 \cdot \frac{3}{2} = 10$.